Topological Invariants and Representation Theory.

This course is a continuation of the course that I gave in the Fall of 2021. Roughly, it will have four parts.

The first part will be an an overview of the material from the Fall 2021. This will take about four first lectures.

Then, in the second part, we will study in depth quantum sl_2, its representations, the braiding for quantum sl_2 and the specialization at roots of unity. After studying the category of finite dimensional representations for quantum sl_2 with divided powers, and the category of modules over small quantum sl_2 we will focus on the corresponding example of a modular tensor category and on its properties.

In the third part I will explain how to construct invariants of 3-manifolds and of 3-cobordisms using modular tensor categories and surgery of 3-manifolds, and I will discuss the general philosophy of topological quantum field theory (TQFT). After this we will focus on how to construct a invariants of 3-manifolds and corresponding TQFT using triangulations of 3-manifolds. After this we will see how these two TQFT's are related.

At the end of the course, in the fourth part, as much as time permits I will explain how these TQFT's constructed combinatorially are related to a quantization of classical Chen-Simons topological field theory and will focus on various conjectures that follow from such (mathematically still conjectural relation).

The **final exam** will be based on take home projects.

Tentative schedule by weeks:

Part one: quantum sl_2.

(1) Overview of Fall-2021 lectures. Invariants of framed tangles as a functor from the category of colored tangles to the coloring category.
(2) Overview of Fall-2021 lectures. Quantum sl_2 as the quotient of the double of its Borel subalgebra. The algebra $U_h sl_2$ over formal power series in h and its universal R-matrix.
(3) Representations of the algebra $U_h sl_2$.
   - Irreducible finite dimensional representations.
   - The decomposition of the tensor product, $q - 6j$ symbols.
   - The R-matrices.
   - Quantum Weyl group.
(4) The algebra $U_q(sl_2)$ over rational functions of q, its integral forms.
(5) Specialization to roots of unity. The algebra with divided powers (we will focus in these lectures only on this integral form). Its finite dimensional representations. Specialization to roots of unity. Irreducible representations, Weyl modules, tilting modules.
(6) The category of finite dimensional modules over $U_q(sl_2)$ at roots of unity, its subcategories of titling and projective modules.The monoidal structure and the braiding.

(7) The quotient M_q of the category of tilting modules over the (ideal monoidal) subcategory of projective modules. Its properties. This category as a braiding fusion category.

Part two: Invariants of 3-manifold via surgery, Reshetikhin-Turaev invariants.

(1) The idea of a topological quantum field theory (TQFT).
(2) Closed 3-manifolds as a surgery along framed link on the sphere S^3. Kirby calculus.
(3) The Kirby calculus for 3-manifolds with boundary.
(4) Invariants of 3-manifolds as invariants of framed links that are invariant with respect to Kirby moves.

Invariants of 3-manifold via triangulation, Turaev-Viro invariants.

(1) 3-manifolds as equivalence classes of triangulated 3-manifolds. q-6j symbols and their properties.
(2) State sum invariants of 3-manifolds of via triangulation (after Turaev and Viro).
(3) The relation between invariants Turaev-Viro and Reshetikhin-Turaev invariants .

Part three: 3D quantum Chen-Simons theory, after Witten.

(1) Path integral quantization of classical topological Chern-Simons field theory (after Witten). How to understand this path integral as a formal semiclassical object. Semiclassical conjecture comparing quantum group construction with the path integral.
(2) Boundary conditions and boundary 2-dimensional conformal field theory (the Wess-Zummino-Witten theory). Concluding remarks.

This schedule may change, depending on how fast the material will go.

Prerequisits Basic familiarity with manifolds, topology, and algebra will be useful.

Some useful references are below. This list is is far from being complete but it gives an idea about the subject. First group of references will help with the material of first two parts of the course. Second group of references is given for the third part.

See also the website of my Fall lectures https://tqfts.com for more reading material. A reasonable degree of familiarity with these lectures will help in the course. I will give more references during the class.

References

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[2] T. Kohno, "Conformal Field Theory and Topology", Translations of Mathematical Monographs, Iwanami Series in Modern Mathematics, AMD 2002.

[3] E. Witten, Quantum field theory and the Jones polynomial, Communications in Mathematical Physics volume 121, pages 351399 (1989).